THE TWENTY-THIRD KELVIN LECTURE.

"THE WORK OF OLIVER HEAVISIDE."

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(*Lecture delivered before* THE INSTITUTION, *21st April*, 1932.)

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(1). THE RANGE OF HEAVISIDE'S WORK.

The Kelvin Lecture of the Institution is usually delivered by an eminent leader in physics, who chooses a subject invariably of great interest to the members, but one not always closely associated with their work. To-night a member of the Institution has been honoured by being asked to give the lecture, and the subject chosen, the work of Heaviside, is one which blends the earliest activity of the Institution, namely, telegraphy, with the latest one, wireless telegraphy and telephony. The subject ranges over the whole of electromagnetic theory and practice, and is too vast to be dealt with in other than a general way.

There is no time to dwell on personal* matters concerning the life and character of Heaviside. I must, however, mention that Heaviside was born in 1850, and that he died a few years ago at the age of 75. His family was eminently telegraphic. He was the nephew of Sir Charles Wheatstone, the inventor of telegraphy. One of his brothers, Mr. A. W. Heaviside, was high up in the telegraphic profession. He himself was engaged for some years in the actual practice and development of telegraphy. All his early papers, from 1872 to 1882, deal with normal problems in telegraphy, such as induction balances, duplex telegraphic circuits, and so on. He appears to have been closely associated with the introduction of quadruplex telegraphy. He retired from practice early, but continued throughout his life to work at telegraphic problems.

Heaviside exemplifies a rare case of the combination of great theoretical and mathematical powers with a bias of mind which was strongly practical. He was entirely self-trained. He found that his work needed mathematics, and he trained himself to be a mathematician. He made himself a physical theorist for the same reason. He was, however, chiefly interested in the practical aspect of signalling problems. He regarded all theoretical work as subsidiary. He was a mathematician at one moment, and a physicist at another, but first and last, and all the time, he was a telegraphist. All his early papers up to 1882 dealt with telegraphic problems by normal methods; but after that date, that is from the age of 32 onwards, he dealt with these problems entirely from the point of view of waves, and therefore as problems in directed radio-telegraphy. Heaviside was the first radio-telegraphist.

His work consists of two main parts: the simplification of Maxwell's theory and the improvement of mathematical methods for use with it; and the design of a transmitting line or cable needed to convey electrical signals perfectly at high speed.

Before dealing with either of these, something must be said about the clearness with which he could express his ideas, either in the English language, or in the language of mathematics. He was a writer of good, vigorous English, often brightened with touches of humour. He chose his words well, and is noted for his resource in suggesting excellent new words to express novel technical quantities coming into prominence. The younger electrical men of to-day hardly realize how many words and phrases now in standard use originated with Heaviside.

A good instance of his clear writing will be found in the first volume of "Electromagnetic Theory," in the form of a chapter of 150 pages descriptive of waves. It is almost free from symbols. A few equations are given to help the wording to make the exact meaning clear, but not such as to require the reader to use any analytical skill. It is the best chapter yet written on electrical waves, and should be read by all interested in wireless work.

Heaviside's descriptive power had a great influence on his life work, since it was his only support in his effort to convince the world that the self-induction of the line was "the long-distance telephoner's best friend." In dealing with his cable problem he laboured under every conceivable disadvantage. He was working entirely alone. He had to solve a mathematical problem so new and difficult that no existing form of analysis was suited to deal with it. He evolved an experimental process which found no favour with mathematicians. It was something which nobody could understand, and which Heaviside himself did not profess to explain. Physicists were keenly interested in Heaviside's work on electromagnetic theory, but naturally left to technical men any applications to practice. The only men whom Heaviside could hope to interest were telegraphists, and in regard to them Heaviside had to struggle against a fixed erroneous belief of the whole profession. Lord Kelvin's pioneer work on ocean telegraphy was highly

* These have been dealt with in an interesting way in Appleyard's "Pioneers of Electrical Communication" (Macmillan and Co.).

scientific and original. It also had the engineering characteristic, that it was made as simple as possible by taking note of every essential point while neglecting anything which did not appreciably affect the result. The self-induction of the line was ignored, and a working principle was reached which came to be known* as the "*KR* law." For Kelvin's purpose the neglect of self-induction was quite justifiable and in every way admirable; but as signalling at higher and higher speeds became a want, and as the character of the signals altered from the simple dots and dashes of telegraphy to the highly complex signals of telephony, the *KR* law of Kelvin ceased to apply because it was no longer sufficiently comprehensive. Heaviside advocated the increase of the self-induction of the line, an idea utterly opposed to the prevailing notions of the time. Heaviside's method was obscure, while his formulæ were complex. If he had not possessed the power of making his views clear in simple language, I doubt very much whether during his lifetime the loading coil would have been invented.

He had many friends in the telegraphic profession, and that profession has never lacked men of great scientific ability capable of realizing and of making clear to their less-trained colleagues the significance of Heaviside's results, and that these results were not merely the expression of personal views but were obtained by methodical mathematical argument. Such men are not necessarily found in the most important professional positions,† so as to be able to exert enough influence on a Government department reluctant to try expensive experiments. Heaviside never had the chance to put his theories to the test. He did suggest practical ways of increasing the self-induction of the line several years before they were actually tried in France and in America. His suggestions, after some preliminary experimenting, would have developed into an invention. As things turned out, the loading coil had to be worked out by other hands. Great credit is due to the inventor of that coil, and also to those concerned with the more recent method of coating the wire with permalloy, or mumetal; but Heaviside was greater than the inventor, since he pointed out in advance what the inventor had to invent. As a rule, theory follows practice, explains it, and makes large-scale developments possible. In this case theory had to come before practice. It is hard to see how the loading coil could have developed out of trials made in the dark.

It is needless in addressing the members of the Institution to take up time in explaining the loading coil, or in urging its importance. It is enough to say that wherever telegraphy is known the name of Heaviside is held in honour. It would also be hopeless to attempt in a public address to discuss Heaviside's mathematics in detail, but the general character of this work must be referred to. No other mathematician has striven so hard to make his work simple, short, and direct; to avoid wandering into side issues, and to refrain from a display of mathematical fireworks. He showed great care in the choice of symbols, and in the modes of printing them. From this point of view his work is an admirable example of plain English in mathematical form. Important variables are chosen with simple plain shapes, free from curls and thin lines, and suitable for compact assembly in a formula. Associated constants are grouped as much as possible, and are represented by less-prominent letters. The effect is to make the working clear, and to reduce to a minimum the number of symbols used, and also the resultant call on the memory. What happens in technical publications is usually the exact opposite to this, in spite of the fact that there is special need in engineering work to make the mathematics as clear and simple as possible.

I shall illustrate these points by an example well known to all of us, and, since this is the only example which I propose to introduce into the Lecture, I shall use it also to picture the operator method. I shall thus use the example to illustrate both the clearness and the obscurity of Heaviside's mathematics. His analysis when dealing with Maxwell's theory involved advanced and difficult mathematics, but this is not the part of his work which is said to be obscure. The part which is called obscure is his operator method which uses little more than simple algebra.

Heaviside chooses, of course, a suitable shape* for his symbol for current, and writes the law of rise of current in an inductive circuit as

$$RC + L\frac{dC}{dt} = E \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or as

$$RA + L\dot{A} = E$$

Having written down the differential equation, the first thing he does is to discard the calculus, by using Boole's symbol p for the differentiator. He writes the equation in operator form as

$$(R + Lp)\, C = E \quad . \quad . \quad . \quad . \quad . \quad (2)$$

He next simplifies as much as possible the mode of denoting the engineering constants R, L, and E, by using a for R/L and C_0 for E/R, leading to

$$(1 + p/a)\, C = C_0 \quad . \quad . \quad . \quad . \quad (3)$$

Next, in a way bolder than Boole, he proceeds

$$C = \frac{1}{1 + (p/a)} C_0 = \frac{a}{p} \cdot \frac{1}{1 + (a/p)} C_0 \quad . \quad . \quad (4)$$

$$C = \left(\frac{a}{p} - \frac{a^2}{p^2} + \frac{a^3}{p^3} - \ldots\right) C_0 \quad . \quad . \quad . \quad (5)$$

* The law is called by its historic name. K refers to capacity and R to resistance.

† In one of Heaviside's notebooks now in the possession of the Institution, I found the following lines —

"Self-induction's 'in the air'
  Everywhere, everywhere.
Waves are running to and fro.
  There they are, there they go.
Try to stop 'em if you can
  You British Engineering man!

Conceive him (if you can)
  The Engineering man."

At this point some personal feeling seems to have upset the balance of the bard's rhythm. Other lines are omitted. The "man" may possibly have been very fond of boasting that he "sat at the feet of Faraday." Heaviside remarked that "beetles could do that."

* The International Electrotechnical Commission has settled that the letter I is to be used as the symbol for current. But what is the letter I? No one has decreed how it shall be shaped or printed or pronounced. It appears in many forms, none suitable for assembly in a formula, and the most insignificant of all these forms, a little worm rampant, is the one favoured by the electrical engineer. Its most obvious part is a dot which should not be there, since dots should be reserved for Newton's notation for differentiation.

He now applies his " rule "

$$\frac{1}{p^n} = \frac{t^n}{n!}$$

giving

$$C = C_0\left[a\frac{t}{1} - a^2\frac{t^2}{2!} + a^3\frac{t^3}{3!} - \ldots\right] \quad . \quad . \quad (6)$$

or

$$C = C_0(1 - e^{-at}) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

so that after reintroducing the engineering symbols he gets finally

$$C = \frac{E}{R}(1 - e^{-Rt/L})$$

This has been given at length to show the steps. The calculus equation (1) is changed to the operator equation (2) or (3). The operator is shown in (4) in a closed form, and in (5) when expanded as an infinite series of powers of p. The " rule " is then used to convert to (6), an infinite series in powers of t, which in many cases can be simplified by using a known mathematical function as in (7). The operator in this case is a simple one, but Heaviside treats all operators, however complicated, in the same way, and also applies the same rule, so that the example typifies his method in general. The expansion of the operator, and the use of the rule, are obscure steps, but the actual working is always simple. This example we shall discuss later. We must now consider the first main part of Heaviside's work, electromagnetic theory.

(2). Electromagnetic Theory.

I shall confine myself in this Lecture to Maxwell's theory as applied to ether free from matter. If we cut out from the theory all the laws concerned with matter, the laws which remain are few, simple, and fundamental. Heaviside showed that they could be put in the form of four pairs of laws connecting four pairs of physical quantities, each pair showing perfect symmetry as regards electricity and magnetism. In order to bring out the symmetry I shall use the same symbol, with a distinguishing suffix e or m, to denote the two members of a pair.

The quantities are

Quantity	Electric	Magnetic
Force intensity	$\mathbf{H}_e$	$\mathbf{H}_m$
Flux density	$\mathbf{B}_e$	$\mathbf{B}_m$
Inductivity	e	m
Energy density	T_e	T_m

The four pairs of laws consist of three which are merely descriptive of permanent relations between the above physical quantities, and of a fourth which states the important working laws controlling the changes that take place. They are:—

Law 1. The force-flux law

$$\mathbf{B}_e = e\mathbf{H}_e \qquad \mathbf{B}_m = m\mathbf{H}_m$$

Law 2. The law of continuity

$$\text{div. } \mathbf{B}_e = 0 \qquad \text{div. } \mathbf{B}_m = 0$$

Law 3. The law of energy density

$$T_e = \tfrac{1}{2} H_e B_e \qquad T_m = \tfrac{1}{2} H_m B_m$$

Law 4 (c.c.). The law of cross cutting

$$\left.\begin{matrix}\mathbf{H}_e\\ \mathbf{H}_m\end{matrix}\right\{ \begin{matrix}\text{is the rate at}\\ \text{which the lines of}\end{matrix} \left.\right\} \begin{matrix}\mathbf{B}_m\\ \mathbf{B}_e\end{matrix}\left\{\begin{matrix}\text{cut unit}\\ \text{length along}\end{matrix}\right\} \begin{matrix}\mathbf{H}_e\\ \mathbf{H}_m\end{matrix}$$

The quantities e, m, are constants, T is a scalar variable, $\mathbf{H}$ and $\mathbf{B}$ are vectors. $\mathbf{H}$ is reckoned per unit length, $\mathbf{B}$ per unit area of cross-section, and T per unit volume. A vector is denoted by a letter in heavy type, and its tensor by the corresponding letter in italic type. $\mathbf{B}$ is the density of Faraday lines or tubes. The lines $\mathbf{B}_m$ are due to a distant magnet or magnetizing current, and the rate of cutting of $\mathbf{B}_m$ is $\mathbf{H}_e$, the electric force, or e.m.f. per unit length. The lines $\mathbf{B}_e$ are due to distant electric charges whose movements represent one or more currents involving corresponding movements of $\mathbf{B}_e$. The rate of cutting of $\mathbf{B}_e$ is $\mathbf{H}_m$, the magnetic force, or m.m.f. per unit length. In the language of action at a distance, $\mathbf{H}_m$ is said to be due to a remote magnetizing current. In the physical language of Faraday it is due to the local cutting of $\mathbf{B}_e$.

There is much to be said about these laws, but I have only time now to mention some of the points connected with Heaviside's work.

In the first law e and m are not mere numbers. $\mathbf{B}$ and $\mathbf{H}$ differ in physical nature as much as an ampere does from a volt. A current and a voltage may be made numerically equal by adopting suitable units, but no choice of units can possibly make an ampere the same kind of thing as a volt. Heaviside's book is the only one on electromagnetic theory in which the distinction between $\mathbf{B}$ and $\mathbf{H}$ is always kept clear.

Heaviside uses, instead of $\mathbf{B}_e$, Maxwell's displacement symbol $\mathbf{D}$, but points out clearly that " the true analogue of $\mathbf{D}$ is $\mathbf{B}_m$."* Heaviside's $\mathbf{D}$ is a multiple of Maxwell's $\mathbf{D}$, since he adopts what he calls a " rational " system of units, the great advantage of which is that all the important laws can be expressed without containing a needless factor 4π, or $1/(4\pi)$, so that the statement of the laws becomes simpler. In ordinary books the expression for T in law 3 contains a factor $1/(8\pi)$ instead of the factor $\frac{1}{2}$.

The most important point refers to the law of cross cutting, 4 (c.c.). This is not expressed as Heaviside put it. It is stated in a way to show its physical aspect. It needs a little more to make it strictly mathematical. The addition can be made in more than one way. We may express the law in local-action form, or in remote-action (and sum-total) form, involving the idea of action at a distance. This can be illustrated by law 3. The energy density $\frac{1}{2}H_eB_e$ is the local part of the total energy $\frac{1}{2}V \times KV$, where V is the potential between the plates of a condenser whose charge is KV. The Faraday tubes connecting the plates spread out through the whole of space. Maxwell proved that the quantity $\frac{1}{2}H_eB_e$ per unit volume summed up for the whole of space was equal to $\frac{1}{2}KV^2$. He proved a similar relation in connection with the companion law for magnetic energy. Exact ideas about energy are largely due to Kelvin, Helmholtz, and Joule, while most of the laws of energy in sum-total form are due to the work of Kelvin. The principle of the localization of energy is, however, due to Maxwell, and forms one of the two important contributions made by him when developing Faraday's ideas into the electromagnetic theory of light. The other new principle was the concept of dielectric current as the rate of change of displacement $\mathbf{D}$ or $\mathbf{B}_e$. Thus, in dealing with

* " Electromagnetic Theory," vol. 1, p. 105.

law 4, we must remember that the rate of change of $\mathbf{B}_e$ means current density. Now consider the cross cutting law 4 (c.c.). This in its magnetic form is the basis of electrical engineering, and is well known both in its local-action and in its remote-action form. One is the law of the dynamo armature represented in Fig. 1, and the other that of the transformer denoted by Fig. 2.

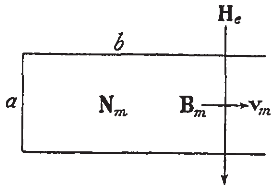

FIG. 1.

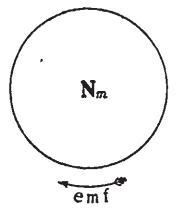

FIG. 2.

$$\text{e.m.f.} = -\dot{\mathbf{N}}_m$$

$$ab\,\mathbf{B}_m = \mathbf{N}_m = \iint \mathbf{B}_m d\sigma$$

$$\mathbf{H}_e = [\mathbf{B}_m \mathbf{v}_m] \qquad \int \mathbf{H}_e d\lambda = \iint(-\dot{\mathbf{B}}_m)d\sigma$$

There is no need to explain what is so well known. The point to note is the great physical difference between two laws which are mathematically equivalent. The first gives $\mathbf{H}_e$, a local e.m.f. per unit length, in terms of the movement at velocity $\mathbf{v}_m$ of a local flux density $\mathbf{B}_m$. The second gives a non-localized total e.m.f. in terms of the change of a distant flux $\mathbf{N}_m$.

The former can be expressed as a vector product

$$\mathbf{H}_e = [\mathbf{B}_m \mathbf{v}_m]$$

The latter can be expressed as

$$L_e = \dot{S}_m$$

where L_e is the line integral of H_e round the circuit, and $\dot{S}_m$ is the surface integral of $(-\dot{B}_m)$ over the area bounded by the circuit.

Heaviside, by taking this circuit infinitesimally small, converted the remote-action law into a local-action form and wrote it

$$[\nabla \mathbf{H}_e] \equiv \text{curl } \mathbf{H}_e = -\dot{\mathbf{B}}_m$$

We need not consider here the full meaning of this mathematical law.

In the magnetic case there is direct experimental evidence for each form of the law, but in the corresponding electric case, obtained by interchanging the suffixes e and m, the experimental evidence available only applies to the law in remote-action form

$$L_m = \dot{S}_e$$

where L_m is the line integral of the magnetic force $\mathbf{H}_m$ round the circuit, that is the total m.m.f.; and where $\dot{S}_e$ is the surface integral of the current density $\dot{B}_e$ over the area bounded by the circuit, that is the total current represented by "ampere-turns." This is the well-known law of the magnetic circuit, a remote-action law. The corresponding local-action law, though not yet established by direct experiment, is deducible mathematically from the remote-action law in exactly the same way as in the former case. Heaviside was the first to notice the extraordinary fact that Maxwell overlooked this point and was content to leave his laws in the language of action at a distance. Maxwell, with a great ideal before him, took up the ideas of Faraday, examined a chaos of known theories and laws, separated out those which fitted in with Faraday's views from those which did not, and developed the former into his great theory of light. Nearly all these laws were expressed in the language of action at a distance. Maxwell could use them, and was content to use them, in that form. It was Heaviside who introduced what he called the second circuital law in local-action form

$$[\nabla \mathbf{H}_m] \equiv \text{curl } \mathbf{H}_m = \dot{\mathbf{B}}_e$$

In doing this he cut out from Maxwell's theory some points which he regarded as not essential, and finally arranged all the fundamental electromagnetic laws in pairs, each of which showed perfect symmetry in regard to electricity and magnetism.

The second circuital law has an equivalent vector form

$$\mathbf{H}_m = [\mathbf{v}_e \mathbf{B}_e]$$

This form of the law was, in essence, recognized both by Heaviside and by Poynting, but neither of them used it in analysis.

The working law of cross cutting [law 4 (c.c.)] can thus be expressed in three equivalent ways, as follows:—

(i) the experimental law in remote-action form as used by Maxwell

$$\int \mathbf{H}_e d\lambda = \iint(-\dot{\mathbf{B}}_m)d\sigma \quad . \quad . \quad . \quad . \quad (4x)$$

$$\int \mathbf{H}_m d\lambda = \iint(+\dot{\mathbf{B}}_e)d\sigma$$

(ii) the mathematical local-action law which Heaviside used and called the circuital law

$$[\nabla \mathbf{H}_e] \equiv \text{curl } \mathbf{H}_e = -\dot{\mathbf{B}}_m \quad . \quad . \quad . \quad (4c)$$

$$[\nabla \mathbf{H}_m] \equiv \text{curl } \mathbf{H}_m = +\dot{\mathbf{B}}_e$$

(iii) the physical local-action law which appears to state exactly the views of Faraday* when expressed in vector form

$$\mathbf{H}_e = [\mathbf{B}_m \mathbf{v}_m] \qquad \mathbf{H}_m = [\mathbf{v}_e \mathbf{B}_e] \quad . \quad . \quad . \quad (4v)$$

Much was done between 1883 and 1885 by Heaviside and by Poynting, working independently, to make clear and to extend Maxwell's theory. Time will not allow of my discussing it in this Lecture, but I have added an historical Appendix dealing with it. The main point to note now is that Heaviside, after arriving at the working law 4 (c), developed from known mathematical theorems an easily workable form of vector algebra particularly suitable for use in connection with it. He did not add anything fundamentally new to Maxwell's theory, or anything in analysis really new mathematically. His work on electromagnetic theory offers a great contrast to that connected with his operator method. Mathematicians have had nothing to say against the one, and nothing to say in favour of the other. They have been very silent on the whole matter.

* "Experimental Researches," vol. 1, p. 529, § 1658.

What Heaviside did was to make the theory so workable that it had the properties of an automatic machine. This sometimes needed the skill of a Heaviside to set it to its task, and sometimes needed the driving power of a Heaviside to make it go. All the same, it was automatic. One end was fed with a mixture of facts and fancies about matter, and out from the other came the results, not as single spies but in battalions. Many of these results were of great interest to the scientific world, and most of the leaders of physics, from Lord Kelvin downwards, discussed them with Heaviside, either personally or through the medium of the scientific Press.

All these problems involved the action of matter, the properties of which are a never-ending source of controversy. This was bad enough in Heaviside's time. It has become far worse since. Nowadays the properties of matter seem to alter every fortnight. Heaviside's results and views were always criticized, but were always treated with great respect. In 1891, Heaviside summed up his work on Maxwell's theory in a single paper printed by the Royal Society in 1892. This was the most important and the most ambitious paper Heaviside ever wrote. It is fairly safe to say that no one yet born has been able to understand it completely. This assertion would be a bold one to make were it not for the fact that there is the great authority of Lord Rayleigh to support it. A letter, now in the possession of the Institution, was written to Heaviside by Lord Rayleigh as Secretary of the Royal Society, when accepting the paper on behalf of the Society, on the 31st October, 1891. To this letter a personal note is added which reads:—"Both our referees, while reporting favourably upon what they could understand, complain of the exceeding stiffness of your paper. One says it is the most difficult he ever tried to read. Do you think you could do anything by illustrations or further explanations, to meet this? As it is, I should fear that no one would take advantage of your work."

The obscurity of the paper does not arise in the mathematics itself, since this, though having a strong Heaviside flavour, is all of standard type and can present no difficulty to the professional mathematician. It is far otherwise with the assumptions underlying the mathematics. These involve numerous physical conceptions for the most part outside the range of experimental evidence, and so abstruse and so little recognized that, however expressed in words, they must be left in such a state that they can be readily misunderstood. The reader can be excused if he fails to follow them. Yet Heaviside was so consistent a man that, if his ideas could be definitely interpreted, his scheme, modified as found necessary, would probably form a firm framework on which permanent physical theories could be built. As it stands, the paper is quoted but rarely, if at all.

We must, however, leave pure theory and proceed to consider Heaviside's application of it to his telegraphic problems.

(3). Radio-Waves and Matter.

Heaviside's object was to design a telegraph cable to transmit signals perfectly at high speed. It is commonly supposed that he solved this problem by means of his operator method. This is only partly true. He used arithmetic as much as algebra, and he used physics more than either. His mathematical result was always an infinite series. This was but natural. It is as unreasonable to expect a mathematical solution to be a simple expression, as to expect the answer to a numerical question to be a simple number such as 3, 4, or 5, instead of a long series of decimals. Solutions given in terms of functions, or of integrals, are all infinite series in disguise. Such solutions are convenient if the functions, or integrals, can be got from tables; otherwise the apparent simplification is little more than a sham. Few seem to realize the large amount of drudgery in numerical calculations which Heaviside went through in order to grasp the meaning of the various forms of infinite series met with in his analysis, or that he made use of every physical argument which he could think of in order to interpret his result. The solution of his problem is to be found, not in his formula, but in his essays. It was by means of these clearly written essays that he at last succeeded in convincing a reluctant telegraphic profession of the truth of his ideas.

He had necessarily to express his problem in terms of coils and condensers and resistances, but he was always thinking in terms of waves. We have thus to consider what the laws have to say about waves.

The laws we need are:—

Law	Electric	Magnetic	
Force-flux	$\mathbf{B}_e = e\mathbf{H}_e$	$\mathbf{B}_m = m\mathbf{H}_m$	. . (1)
Energy	$T_e = \frac{1}{2}H_eB_e$	$T_m = \frac{1}{2}H_mB_m$	. . (3)
Cross-cutting	$\mathbf{H}_e = [\mathbf{B}_m\mathbf{v}_m]$	$\mathbf{H}_m = [\mathbf{v}_e\mathbf{B}_e]$	. . (4v)

We postulate that (i) there is no such thing as action at a distance, and (ii) light waves do not interfere with each other. Light waves received from a particular star are not affected by light emitted by other stars and passed on the journey from the special star. When we look at a star we assume that we see light from that star only, and not a mixture of light from all the stars. The programme of a particular broadcasting station can be reproduced perfectly by a suitable receiver irrespective of the activities of other stations. This could not be the case if the waves merged into one another and lost their separate individuality.

Now suppose that from any cause we have a disturbance represented by two values of H and the corresponding values of B. What happens? We have not to consider action at a distance, or even any other disturbance in the neighbourhood. The laws must thus apply directly to our particular disturbance.

If in (4v) we put one of the velocities, say v_m, equal to zero, it will be seen from (1) and (4v) that all four quantities **B**, **H**, vanish. Thus whatever the disturbance may be it must be in motion.*

Next let us assume that it is possible under certain conditions for the electric and magnetic fluxes to move together so that the disturbance moves like a wave of light which remains unchanged as it progresses. We

* This result appears at first to be in conflict with the case of a steady field due either to a permanent magnet, or to a charged condenser. But it can readily be shown that the product of v_e and v_m is always the constant v^2. The two steady cases are the mathematical limits obtained by making one velocity infinite and the other infinitesimal. Thus, if $H_m = M$ and $v_m = cv$, where c and cv are each high infinitesimals, we find $H_e = cv \,.\, mM$, $T_m = \frac{1}{2}mM^2$, and $T_e = c^2T_m$. If $c = 0$, we may have finite values for M and T_m, and yet zero values for v_m, H_e, and T_e.

can find these conditions by putting $\mathbf{v}_e = \mathbf{v}_m = \mathbf{v}$. If no change takes place the two velocities must clearly be the same.

From (1) and (4v) used numerically we find

$$B_e = eH_e = evB_m = evmH_m = emv^2B_e$$

or $$v^2em = 1, \text{ that is } v = (em)^{-\frac{1}{2}}$$

Thus the velocity must be fixed in amount, and from (4v) used vectorially it follows that $\mathbf{H}_e$, $\mathbf{H}_m$, and $\mathbf{v}$, are perpendicular to each other in pairs.

If we use in (3) the value of v found above, we get numerically

$$\tfrac{1}{2}H_eB_e = \tfrac{1}{2}H_mB_m = \frac{1}{2v}H_eH_m$$

This means that the electric and magnetic energy densities are equal, and that they are represented by vector fluxes at right angles to each other, and also to

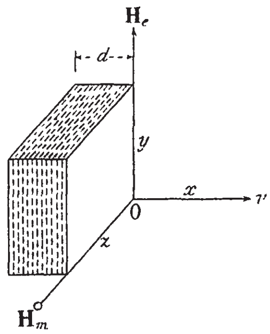

FIG 3.

the direction of the fixed velocity v. These fluxes must bear a constant ratio to each other

$$\frac{\mathbf{B}_e}{\sqrt{e}} = \frac{\mathbf{B}_m}{\sqrt{m}} \qquad \mathbf{H}_e\sqrt{e} = \mathbf{H}_m\sqrt{m}$$

These are the ordinary properties of a ray of light.

For our present purpose the above laws yield another conclusion which is of more immediate use. We are assuming that the disturbance moves as a whole, along the axis of x, with a velocity v. Consider a rectangular block of it with edges each parallel to one of the axes. Let d be the depth of the block along the axis of x, and let the sectional area perpendicular to x be unity. Let $\mathbf{H}_e$ be along the axis of y, and $\mathbf{H}_m$ be along the axis of z. The positive direction of z is upwards from the paper as indicated by Silvanus Thompson's symbol ⊙. Consider the flow of energy, in the path of the advancing pulse, at some point O. This is zero until the pulse reaches O, it has some value **P** per unit cross-section while the pulse passes O, and is zero again after the passing of the pulse. The time taken for the pulse to pass in d/v. The total energy which passes per unit of cross-section must be equal to that contained in the block of volume d, or

$$P\frac{d}{v} = (\tfrac{1}{2}H_eB_e + \tfrac{1}{2}H_mB_m)\, d = \frac{H_eH_m}{v}d$$

Thus P is numerically equal to, and is given in vector form by,

$$\mathbf{P} = [\mathbf{H}_e\mathbf{H}_m]$$

Now this is the Poynting flux, discovered and proved by Poynting in 1884, and constituting the greatest contribution to electromagnetic theory since the death of Maxwell. The above argument does not prove the Poynting flux since we have made an assumption, but from our present point of view the important thing to note is that the value established by Poynting for this energy flux is exactly what is required to justify the assumption made, that an electromagnetic pulse with equal electric and magnetic energies per unit volume, and represented by fluxes at right angles to each other, moves forward at a fixed speed v perpendicular to each flux, carrying all its energy with it, and thus *leaving no energy behind it*. This means that no disturbance can be left in its track, since such a disturbance would need energy, and there is no source of such energy.

This explains the fact, always assumed about light, that a wave of light passes from a star to the earth without changing its character on the journey. Each ordinate of the wave represents a thin pulse of light which travels unchanged without affecting, or being affected by, any pulse which precedes it, follows it, travels with it, or crosses its path.

Heaviside's problem of high-speed telegraphy, including the more complex case of telephony, was that of finding the conditions to be fulfilled by the transmitting line, and by the receiving apparatus, in order that a pulse of potential, applied at one end of the line, would, in spite of the interference of matter, pass along the line, like the pulse in a ray of light, without leaving a trace of energy behind it to interfere with any succeeding pulse.

A large part of Heaviside's work was devoted to the study of reflected waves, but his whole object was to find out how to avoid them. The limit of the working speed of telegraphy is fixed by the fact that the reflected waves distort the signals and mix them up. He showed how to design the transmitting line and the terminal apparatus so that, in essence, reflections could be prevented. " It is possible to absorb completely an arbitrary wave by means of a suitable terminal resistance. Then there are no reflected waves."* When a circuit fulfilled the requisite conditions he called it " distortionless." His discovery of these conditions, and his recognition of the fact that they could be approximately secured in practice, shows that he had not only a scientific, but also a thorough engineering, grip of the cable problem.

His theory of the distortionless circuit dates from not later than June 1887,† but it would be quite a mistake to suppose that he relied on any one method for establishing the necessary conditions. His operator process was only one of these methods. He tried many ways, and was untiring in working out in detail a host of examples in order to prove that he got consistent results. Some one described genius as the power of taking infinite pains. It is usually a power which a genius displays less than any other. In Heaviside's case we have a strong combination of genius and industry. He did not establish his theory by any single demonstration, but as the result of the study, not only mathematically

* " Electromagnetic Theory," vol. 2, p. 74
† " Electrical Papers," vol. 2, pp. 119–168

but also physically, of an endless number of worked-out cases. It is this fact which makes it difficult to give any simple account of his work.

That, among other ways, he did regard his cable problem as comparable with that of a pulse of light can be shown by quotations from his writings.* In reference to a pure plane wave which "just goes on," he says: "To what extent this will continue depends upon the relation the constants of the circuit bear to the distortionless state. When the latter obtains, the wave is transmitted *without spreading out behind*, so that at a time t the initial state, if existent over a unit length, will be found over unit length at a distance vt to the right" but attenuated. "In all other cases there is reflection in transit, and the reflected portions travel back, besides getting mixed up together, thus making a tail."* "The distortionless state forms a simple and natural boundary between two diverse kinds of propagation of a complicated nature, in each of which there is continuous distortion which is ultimately unlimited." "When a circuit has been brought to this (distortionless) state, then arbitrary signals of any size and manner of variation, originated at the beginning of the circuit, will run along it at the speed of light, and in doing so suffer no alteration save a weakening . . . the succession of values will faithfully repeat those at the origin."

(4). Reflected Waves.

Heaviside, in more than one place in his book, points out that the series and shunt resistances of the transmitting line absorb energy from the advancing wave; that each resistance reduces the electric and magnetic energies but in different proportions; and that unless certain relations hold between the various inductance and resistance constants of the line, some of the Faraday tubes will be reversed. This leads to reflected waves.

In a wave pulse, the values of B_e and B_m represent equal energies per unit volume, and thus bear a fixed ratio to each other, so that by adopting suitable units we can use the same number $(a + b)$ to denote each of them. The square of this number in terms of another unit will represent the total energy per unit volume. The value of P, the Poynting flux of energy per unit area, is proportional to the product B_eB_m, so that this product with due regard to sign, will denote P in terms of yet another unit. If now, owing to the action of matter upon the incident pulse, the electric flux $(a + b)$ is partially reversed so as to become $(a - b)$, we get two pulses, one transmitted and the other reflected, as follows:—

	B_e	B_m	Energy	P
Incident pulse	$a + b$	$a + b$	$(a + b)^2$	$(a + b)^2$
Transmitted pulse	$+ a$	$+ a$	a^2	$+ a^2$
Reflected pulse	$- b$	$+ b$	b^2	$- b^2$

The total energy is reduced from $(a + b)^2$ to $a^2 + b^2$, the loss $2ab$ being absorbed in the resistances. One pulse goes forward with energy a^2, and the other backward with energy b^2. Although this is only an illustration, it appears to represent what happens.

The line resistances must reduce the energy of the advancing pulse, but it does not follow that the pulse must split up. The condition that the pulse fluxes move together is that their energies must remain equal to each other, not that they must continue constant in time. Even in a light wave the energies continuously diminish, because, as the wave spreads spherically, the pulse energies are distributed over an ever-increasing area of wave front, so that the energy per unit volume diminishes. The important point is that, though this is the case, the electric and magnetic energies per unit volume diminish in the same proportion, and since they start equal they always remain equal.

The absorption of energy in the resistances causes not one effect but three. Two of these are direct and of no importance. The third is indirect and is all important. The loss of energy must weaken the signal. This is of no consequence since all that is needed is a more sensitive receiver, and electrical apparatus never fails from want of sensitiveness. The heating of the wire gives rise to heat waves. Each electron in the wire insists on blowing its own trumpet. There results a concert of a kind involving "harmony not understood," but it has no connection with the signal and does not affect it in any way. If, however, the energy absorbed in the resistances is withdrawn unequally from the electric and magnetic energies in the pulse these two energies are no longer equal, the equality of the velocities, $v_e = v_m$, no longer holds, and the pulse must break up into parts. Each of these carries the signal with it. The strength of each part must be proportional to that of the original pulse. The signal is represented by the modulation, or law of variation, of the strength of the pulses in the sequence forming the wave, so that each part represents a sequence having the signal imprinted on it. One part represents a forward wave which proceeds towards the receiver, the other represents a wave which goes back. If it always went back it would not matter, since it would never reach the receiver; but, unfortunately, the reflected wave meets another conducting surface, is reflected a second time, and goes forward again, reaching the receiver as an echo at a time subsequent to the forward wave. It thus arrives with another part of the signal so that confusion results at the receiver.

What happens in an ordinary telegraphic line is illustrated in Fig. 4. Suppose that a pulse of potential, impressed on the line by pressing a key, represents a dot signal d_0 travelling along the line at the speed of light in the direction of the arrow, from a transmitting station T to a receiving station R at a distance of 500 km, represented by 0·5 on the diagram. This pulse will splash against R. Its energy will be in part absorbed, but some of it will take the form of a reflected wave which will return to T, where it will give rise to another reflection, which will follow d_0 as an echo d, after a time corresponding with twice the distance between the two stations, or 1 000 km. The echo d_1 will behave exactly like d_0 and hence there will arise a succession of echoes d_2, d_3, etc. These echoes are shown in the figure to the right of d_0 spaced at distances 1 000 km from each other. Of course nothing exists to the right of T. The pulses d_1, d_2, etc., are but shadows forecasting coming events. The event does not exist until the moment the shadow arrives at T, after which it goes on as a real echo and is received as such at R. The effect at R is, however,

* "Electromagnetic Theory," vol. 1, pp. 408–410.

exactly as if d_1, d_2, etc., were real pulses following each other in procession, all at the speed of light and spaced as indicated. In theory the number of these echoes is unlimited, so that some of them come on with, and others actually behind, a dash signal D impressed on the line by the operator at some time interval after the dot signal d_0. The effect is to spread out the dot signal. For the moment we are ignoring attenuation, and we assume that the line itself is perfect, so that reflections only occur at T and at R.

If, for the dot signal, the key is pressed for only 0·0001 second, during which time light travels 30 km, the depth of the d_0 signal will be 30 km, or much less than 1 000 km, the spacing of the echoes, hence all these " d " signals will be separate. If, however, the key is pressed for 0·01 second the depth of each pulse will be 3 000 km, so that the pulses will overlap. Moreover, the line is not perfect, so that reflections arise at every point of the

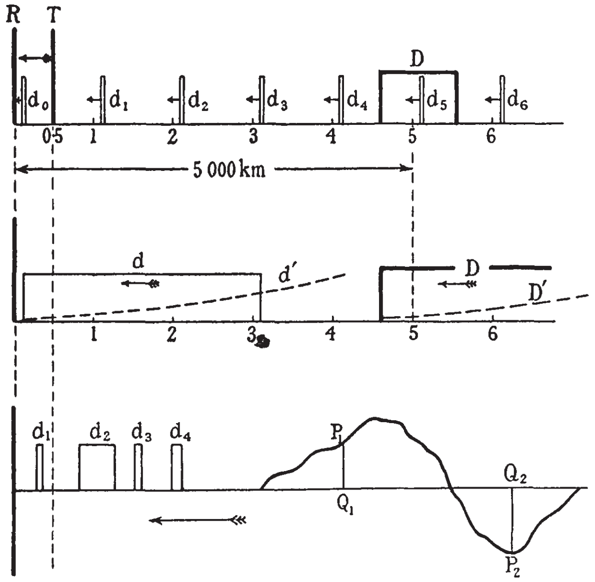

FIG. 4.

path. The result is that, as indicated in the middle diagram, the received signal d′ will not be represented by the rectangle d denoting the signal sent, but by a curve, the ordinate of which, while greatly attenuated at the start, will rise continuously owing to the cumulative effect of successive reflections, and will be spread over a base much greater than 3 000 km, the original depth of the pulse. In theory it will be indefinitely prolonged, and in practice if the dash signal D is made after too short a time interval, the corresponding D′ signal as received, which will also be greatly attenuated at first, will have its early portion overwhelmed by the rear part of the d′ pulse, so that dots and dashes will get hopelessly mixed. If, however, we can prevent the occurrence of reflections, it does not matter how complicated the signal may be. We can assume that it is represented, as in the bottom figure, by an isolated, irregular set of signals d_1, d_2, etc., or by a continuous wave such as P_1P_2 denoting a complex telephonic message. Each ordinate PQ of the wave can be regarded as an independent pulse travelling along the line without interfering with its neighbours, and without leaving behind it any trace of its passage. Attenuation there may be, but it will be the same for each ordinate. The whole wave will go through, keeping its wave-form intact. It only needs a good receiver to interpret the signals perfectly, however rapid and however complex they may be.

Heaviside saw all this clearly before the time of Marconi and before even the earliest experiments of Hertz and Lodge. It is explained in the descriptive parts of his writings, which are devoid of analytical argument. He does not connect up with the equations as I have tried to do. But it is certain that he did this for himself, and that in doing so he used the standard physics of Maxwell's theory, and the standard mathematics natural to it. He did not for this purpose use his operator method at all. This method was not evolved until after he had seen from physics that, in order to make the transmitting line suitable for high-speed signalling certain realizable conditions had to be met. He used the operator method merely to find out these conditions. To do this, he applied main force. The method was an experimental one and needed a thorough mathematical investigation in order to justify it. The conventional mathematician takes much interest in a pure mathematical research, but Heaviside was not a conventional mathematician. He always needed an objective, and, as soon as he saw how to reach it, he went straight for it without troubling more about theory. He was like an engineer who has to reach a result within a strictly limited time, and who has to force the pace and proceed by trial and test, without spending much time on theory.

Before leaving the subject of reflections, reference must be made to the Heaviside layer, since no account of Heaviside's work would be complete without it. The theory of this layer forms one of the most important contributions to the theory of radio-telegraphy. It was Heaviside's recognized position as an authority on Maxwell's theory that drew public attention to the possibilities of this layer, and caused experiments on it to be made as soon as suitable instrumental methods became available. It is well understood now that reflections from the layer enable the waves to keep to the curved surface of the earth. It is found possible to detect these waves after travelling not only a quarter way, or half way, round the earth, but also after travelling once, twice, and even four times completely round the globe.

Prof. E. V. Appleton, whose experimental work on the Heaviside layer is of great interest and importance, has been good enough to help me to illustrate these matters in Figs. 5, 6, and 7. The first two are reproductions of the actual record of receiving apparatus, while the curves shown in Fig. 7 have ordinates representing measured values. Fig. 5 represents round-the-world echoes received on a travelling photographic plate, the time intervals being measured in the ordinary chronographic manner. The emitted signal is double, and the corresponding echoes are distinguished by Roman and Arabic numbers. Each echo arrives about one-seventh of a second after its predecessor, this being the time taken by radiation at the speed of light to go round the earth

by the polygonal path due to successive reflections at the Heaviside layer. Three round-the-world echoes are shown. Fig. 6 represents two signals a, a', received at a German station, from a transmitting station in America. One signal arrives by the shorter route over the Atlantic, and the other by the longer route over the Pacific. In this case the time interval between the two signals is less than one-seventh of a second. The experiments of Figs. 5 and 6 are due to Quäck. The tests shown in Fig. 7 were made by Prof. Appleton, and represent the point, since they prove the existence of the Heaviside layer, and the action of this layer in producing reflections.

(5). THE CABLE PROBLEM AND THE OPERATOR METHOD.

Heaviside realized that what he had to consider was not the behaviour of a complex wave, but that of a single pulse. This pulse is a solitary wave advancing over a dead smooth sea. Heaviside wanted a form of mathematics which would enable him to ride on the front of this wave and to study what happened to it when it

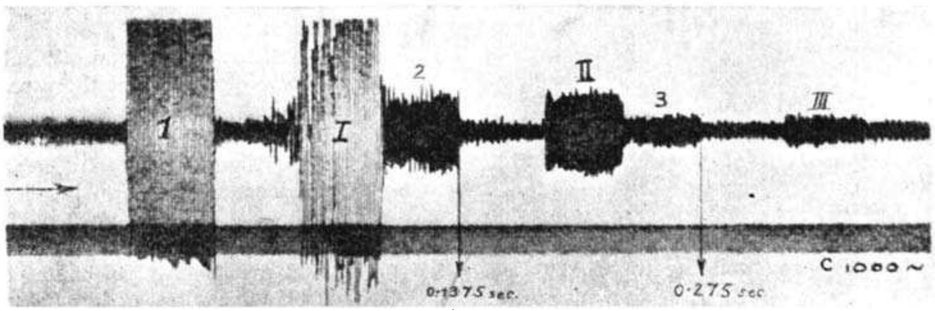

FIG. 5.

record made at King's College of reflections at the Heaviside layer caused by directing a beam of radio waves nearly vertically upwards at the East London College a few miles distant. The waves are reflected in succession at the surface of the layer and at that of the earth, the number of reflections occurring between the two stations being dependent on the original inclination of the particular ray. The figure shows five such met obstacles. In the ether ocean these obstacles are not like rocky islands, or iron-bound coasts. They are much more like floating ships, or floating masses of seaweed. He had to find the conditions under which no reflected waves would occur. It will be found that, though much of Heaviside's mathematics deals with waves, his sole point of interest is what is happening at the wave front. He pays no attention to the rest of the

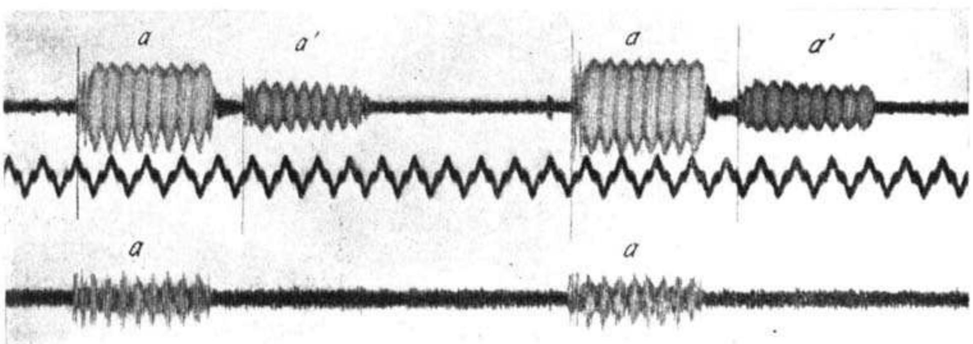

FIG. 6

echoes, F_1 to F_5, besides the direct ground signal G, which arrives first but which is weak in comparison, since the transmitting beam is directed upwards so that its rays are much weaker in the horizontal than in the vertical direction.

Such experiments are all subsequent to Heaviside and we cannot dwell on many points of interest set forth in the original papers.* They are, however, quite to the wave. He merely wants to know what happens to a thin pulse.

It was not for want of searching that Heaviside quite failed to find in mathematical work any form of analysis suitable for his purpose. Those who have implied that what he did with his operators can be done by other and better methods, cannot have understood what the problem really was. There are several methods for dealing with the steady or periodic state, and also for

* For the work of Quäck, see *Jahrbuch der drahtlosen Telegraphie*, 1926, vol. 28, p. 177, and 1927, vol. 30, p. 42. The subject is dealt with by Appleton in a lecture on "Wireless Echoes" before the British Association, 1930, and also in "Wireless Echoes of Short Delay,' *Proceedings of the Physical Society of London*, 1932, vol. 44, p. 76. Appleton and Barnett first proved the existence of the Heaviside layer at a height of 90–100 km, using broadcasting wavelengths (*Proceedings of the Royal Society*, A, 1925, vol. 109, p. 621). Appleton also was the first to find the second or upper layer in 1927 (*Nature*, 1927, vol. 120, p. 330). This upper layer is reached by short-wave radiation after penetrating the lower layer. The tests on Figs. 5, 6, and 7 were all made on the upper layer.

dealing with the building up of such a state by transients represented by exponential changes or undulating surges. Heaviside's operator method can be applied to these,* and, as a matter of fact, in technical publications it has been more used to deal with such states than for Heaviside's main purpose. Heaviside was not in the least interested in either of these states, but in an initial state which preceded both. The mathematician, when dealing with a problem involving distributed capacitance and inductance, divides up the circuit into elements, and uses differential equations in connection with them. But, however small the condenser element may be, it is assumed that the distribution of Faraday tubes connecting the plates is just as if the voltage between them were steady, and as if the tubes extended throughout space. The same assumption is made with regard to the inductances. These assumptions are quite wrong at the start, since a disturbance can only spread at a limited speed v, so that at a time t it must always be confined within a sphere of radius vt. Heaviside was interested only in the front of the advancing disturbance. The transient state represents the building up of the final

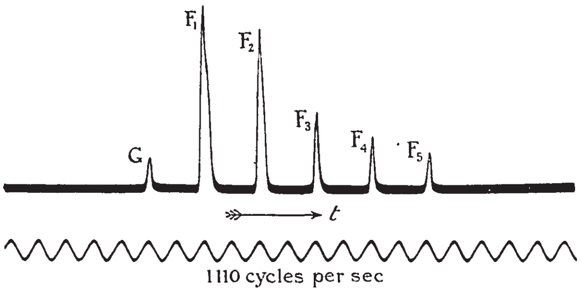

FIG. 7.

state by means of the cumulative effects of a multitude of reflections which are successive, not simultaneous. Heaviside wanted to study the behaviour of the first pulse which travels beyond the reach of even the first reflection. Mathematics contains much about waves. Hertz gave some beautiful examples in one of his papers. Such work either is confined to ether free from matter, or deals with forms of matter so simple mathematically that they do not at all resemble a telegraphic circuit. Heaviside wanted to reach a definite result in an actual engineering problem. He was not the man to replace the physics by a convenient assumption in order to revel in pure mathematics. Instead of discarding the physics, he threw away the mathematics and stuck to his problem. Finding no mathematics to help him, he went ahead without it.

The problem being to find out what happened to a pulse, Heaviside had to do two things. He first had to express a pulse in mathematical language; he next had to deal with the pulse. For the first purpose he used his unit function, and, for the second, his operator method. The two are quite distinct, but since each is used in the same example, while neither is explained, it often seems to be assumed that they are essential parts of one method. They form two incomprehensibles, and the presence of the one is held to account for the mystery of the other. They are quite separate things, one being wanted to formulate the problem, and the other to solve it. Heaviside deals with each of them in a way which shows up his resource and also his attitude towards mathematics. He was more interested in the use of mathematics than in the making of it. He showed this in his work on the cable problem, just as Fourier did in his work on the theory of heat. The work of each has been regarded by the rigid mathematician as sadly lacking in the rigour now "required" by such mathematicians.

Heaviside held that the most convincing of all proofs of a method was the verification of results predicted by it; that there were ways other than mathematical ones for proving things; and that sometimes these other ways could even be used within purely mathematical regions. In particular, since mathematics can often be used to deduce, from known physical results, other results not yet established, it must at times be possible to use the physics of two sets of known results to prove the strength of a mathematical chain connecting them. He frequently uses physical arguments to establish mathematical results "Familiarity with the working

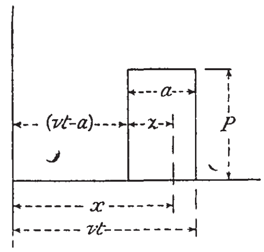

FIG. 8.

out of physical problems breeds contempt for the idea of requiring a special demonstration of what seems to be necessary." He even says: "Physics is above mathematics, and the slave must be trained to work to suit the master's convenience." Heaviside has never been popular with mathematicians.

It is easy to represent by a curve a pulse having strength P for certain values of the variable z, and having zero strength for all other values of z; but it is not easy to find a mathematical formula for it.

Heaviside assumed the existence of a unit function $H(z)$ which was to be unity whenever z was positive, and zero whenever z was negative. A stationary pulse of strength P, and base a, can then be expressed as

$$P\big[H(z) - H(z-a)\big]$$

since outside the range $0 < z < a$ the two values of H are either both zero or both unity, so that their difference is zero and the quantity vanishes; while within this range one of the values of H becomes unity and the other zero, so that the expression gives the true value P of the pulse.

If the pulse is moving with velocity v, it will be seen from Fig. 8 that we have at any time t

$$z = x - (vt - a)$$

and by substitution we see that

$$P\big[H(x - vt + a) - H(x - vt)\big]$$

* The result in such cases is simply to establish the ordinary alternate-current formulæ involving impedances, or complex numbers Heaviside had done all this in 1878 by Boole's method, in his early telephonic papers, before alternate currents had come into commercial use.

represents the true value of the pulse, whatever values we give either to x or to t.

This unit function is used by Heaviside throughout his work. He argued from physical considerations that such a function must exist. He did not know, and did not even try to find out, what it was. He could see that it must have certain simple properties, and that he could use it. This was quite enough for Heaviside. His argument in what he calls " physical mathematics " can be illustrated with the aid of the example already referred to.

Let a switch close at time 0 a circuit governed by

$$E = RC + L\dot{C}$$

with $E = 20$ volts, $R = 5$ ohms, and $L = 5$ mh. The mathematician gives us the solution as

$$C = \frac{E}{R}(1 - e^{-Rt/L}) = 4(1 - e^{-1\,000t})$$

This tells us that C reaches its steady value of 4 amperes in a small fraction of a second. It also tells us that at a time of $-$ 1 second it was (approximately) $- 8 \times 10^{434}$ amperes. Heaviside talks darkly of men who strain hard over a few gnats yet " swallow, quite unawares, *all* the camels in Arabia." Yet what is the act of swallowing a few camels compared with that of swallowing a negative current such as this? Common sense may point out that the switch was not closed at the moment in question. This may be common sense. Is it mathematics? The switch is an important part of the circuit but forms no part of the mathematical problem. The precise mathematician, who declares that mathematics is nothing if not accurate, has given us an answer which is not even approximately correct. He says blandly that t is restricted to positive values. But who has so restricted t, and where does the restriction appear in the mathematics? The fact is that E is not 20 volts, but 20 $H(t)$ volts, or alternatively that R is 5 ohms divided by $H(t)$. The factor $H(t)$ appears in the answer for C, as well as in the original equation. This is the mathematical reason why C is zero until t is positive.

This example can now be used to express in full form the operator problem, which we see involves not only the operator but also the special function $H(t)$. We have

$$C = \frac{1}{R + Lp} E \, . \, H(t)$$

This is only a simple case. Let us consider what it means mathematically. The $H(t)$ function, which Heaviside did not even try to formulate, can be shown to be an infinite series of a most complicated kind. The pulse E, to which the operator is applied, may in simple cases be a constant, but in general it changes in an arbitrary fashion over a range of the variable, so that it has to be represented by a Fourier series also containing an infinite number of terms. The operator itself, whether simple or complicated, is always expanded by Heaviside into an infinite series of terms involving powers of p. Heaviside did not know, and did not try to find out, what p was. He merely knew how to use it. We have thus to deal with the product of three infinite eries.

A mathematician will discuss at length the convergency or divergency of an infinite series, and at even greater length will point out the grave risks attending any attempt to multiply together two such series. His views about the multiplication of three series, like these of Heaviside, do not seem to have been published. There may be limits to what a publisher can print.

Heaviside's view of the problem was not that of a mathematician but that of an engineer. His problem, however formidable in appearance, is, after all, much like that which the ordinary engineer has to face every day in his life. Consider how an engineer deals with a substance like steel. Any good modern book on physics will convince you that it is appalling to contemplate the multitudinous things about the structure of steel which the engineer does *not* know. What should give rise to no little surprise is that, in spite of such a terrible curse, not one of them seems a penny the worse. The engineer does not, to begin with, try to find out all about steel. He simply uses steel. He tests it, finds out some of its properties, learns from experience that he can rely on those properties, designs his engines and structures accordingly, and gets results quite satisfactory, not only to himself but to the world at large. No one is so sure as the engineer that, if a steel will stand a pull of 10 tons to the square inch, it will go on standing that pull in spite of all the structural theories depending on groupings of molecules, or of atoms, or of electrons, or of any of the other " ons " by means of which modern physics explains, or explains away, the structure of steel.

Heaviside's motto was "first get on in any way possible, leave the logic for later work." He got down to his problem and said to himself something to this effect:—I have now reached at last an exact mathematical representation of my problem . . . well . . . if this is a precise expression of the problem to be solved, it must in some way be an equally precise statement of the answer to be found. There is no need to worry any longer over mathematics. I have simply to read off the answer.

He tried in all sorts of ways to do this. The marvel is that he got through. He no doubt had failures, and no doubt used these as steps to success. He soon found out how to intepret his operator p so that in some cases, in which the solutions were already known, he got the right results. He thus arrived at his " rule " by use of which he could get a definite result in every case. These results proved to be consistent in connection with similar problems, or with the same problem in various forms. Heaviside knew well, and admitted fully, that such results were not proved. No one troubled less about proofs, but no one troubled so much about verification. He proved nothing. He simply made sure he was right. With Heaviside a proof was in general little more than a way of meeting the whims and fancies of other people. To make sure of being right is to appeal to experience for a judgment. This test Heaviside never failed to apply in ample fashion.

Now we all use experience, but theorists are apt to forget that on all questions experience is the final court of appeal. The engineer makes good use of matter without knowing what it is. He reproduces results, and forms his own theory about the process. There never

was a greater mistake than to suppose that the practical man looks down on theory. No one thinks so highly of it. He only makes one condition. The theory must be his own. In this respect the engineer is sadly like men who are not engineers. The theory may be crude, yet it is founded on experience, and, within the range of that experience, it is reliable. That is enough for the engineer.

Next take the physicist. No one evolves such weird theories about matter. They make one ask:—Why do these imaginations rage so furiously together? Yet, however wild the theory, if it leads to experiments yielding consistent results, and if these results serve to connect facts not previously associated together, the physicist is quite sure that he has made an advance, and that his theory contains some of the truth. No theory can contain the truth, the whole truth, and nothing but the truth.

Finally, take the mathematician. What does the mathematical leader do when he is in difficult case? He evolves what he calls a symbolical method. He does not explain, or even define it. He is not quite sure of his ground, but he sees a possibility of advance. In essence he proceeds just as Heaviside did, though he is rarely so frank about it. He does not understand his method, but goes ahead as if he did, and tests his results for consistency. Now consistency of results is to the mathematician what experimental tests are to the physicist and what experience is to the engineer. It is the mode of testing the range within which the method can be trusted to give reliable results. In no case is a full proof given.

If there is any difference between the unrecognized method of Heaviside and what is one of the standard methods of the pure mathematician, it is due to a fact characteristic of Heaviside. In his case there was always a definite object in view. From this he never turned aside to evolve interesting mathematical theorems. He worked steadily on at his actual problem until he carried it through to a finish. The formula was not the finish, as is usually the case. With Heaviside the formula was but the beginning, not the end, of the solution. He went to great labour with arithmetic to grip the meaning of his formula, and used every physical argument he could think of to realize its application. His solution was reached more by physics than by mathematics. The most extraordinary fact about the operator method is that it is an abstruse form of analysis which might have been expected to attract the mathematician and to repel the technical man, yet it is to technical publications one must turn for appreciation of Heaviside's work. This work aimed at useful results, and it is these that appeal to the engineer. It was the useful trend of the work that deflected from it the attention not only of the mathematician but even of the physicist. The physicist did not realize that, technical as it was, the problem was in the main one in pure physics, and the mathematician did not see until quite recently that the analysis in spite of its usefulness was yet a form of mathematics of singularly pure type. Since Heaviside did not profess to explain his method, it would be out of place in an account of his work to dwell upon what he did not do; but it would not be right to make no reference whatever to the large amount of mathematical work which has been done in making clear the actual working of the method and in searching for a firm base on which to justify it.

The late Prof. Bromwich was the first mathematician of high standing to show sympathy with the operator method. His earliest paper was published in 1916. Since Heaviside's death in 1925 other efforts at explanation have been made, and several good books dealing with the method have appeared in print. These books explain in a simple way the actual working of the method in connection with examples such as Heaviside used, and also in regard to what may be called more up-to-date examples in telegraphy. Whether any fundamental theory of the method is given is, however, quite another matter.

The two most original contributions to this subject appear to be those of Prof. Bromwich and of J. R. Carson. Bromwich's method is one of great beauty and power. It justifies and extends Heaviside's results. Nevertheless, I cannot imagine two methods more unlike than those of Bromwich and of Heaviside. The Carson method, on the other hand, is very similar in form to that of Heaviside. It approaches the subject from quite an independent starting point, and, by basing the analysis on what is known as an integral equation it puts to new use a large number of established mathematical results in a way such as greatly to extend the application of the method. Moreover, Carson applies Borel's theorem. By so doing he extends the method still further and, what is even more important, he appears to give the method much additional power. This has been shown in the work of Carson himself, and still more in quite recent work by Balth van der Pol.

In my own contribution to the subject the aim has been to formulate the unit function in precise mathematical language, and also to establish theorems justifying the use of Heaviside's index operator p. In my view, the connection between the methods of Heaviside, Bromwich, and Carson, is simply this. Each is a symbolical method. Every symbolical method proceeds on assumption, since there must be an assumption about any method which involves taking a step that is not fully explained and justified. Each method starts from a different base, but the actual working is not questionable in any case; the results reached in each case are the same, and are also known to be correct. The natural conclusion is that there is something common to the assumptions made in the three cases, and that this common part must be true. I have explained my views of this elsewhere, but, whatever the truth may be, the great interest of Heaviside's work does not lie in any theoretical justification of it which has been, or which may be, given, but in the fact that Heaviside got through without any explanation at all. He evolved a working method based upon trial, confirmed on test, and finally put to use on the largest scale humanly possible.

Conclusion.

Heaviside's work can be summarized in four conclusions, three of which are not likely to be disputed, while the fourth is, from its nature, very debatable and will certainly not be generally admitted.

In the first place, Heaviside's actual analysis was always simple, clear, direct, and compact. His writings do not exhibit the orderly arrangement of a textbook. It was his characteristic never to leave a definite problem unfinished; but when engaged on pure theory he could hardly get through a page without wandering off into some application to physics involving more or less debatable assumptions about matter. He never troubled to assemble his results. He uses his operator throughout his work, but does not trouble to give anywhere a complete definition of it. He puts the electromagnetic laws into the simplest and most symmetrical of forms, but nowhere collects them into four lines, as can readily be done. He was not writing a textbook for beginners, and certainly he was not writing a so-called elementary treatise for those who love rigour. He was dealing with electrical problems in a series of articles published in the *Electrician*, a weekly technical journal. The articles were spread over a period of many years. He was allowed great freedom as to what he wrote, but one condition was always strictly imposed. After first publication not a letter could be changed, since the type was kept set up until the corresponding section of the book was printed. Whose writings would survive so well as those of Heaviside a test so severe and so prolonged? These writings are often called obscure. This is true in two important respects. His efforts to make things clear seemed to stop with himself, for he never seemed to imagine that a point clear to himself could be other than clear to his reader. He would at times stride casually over a mountain and leave his reader to struggle after him to reach a ridge from which he could be seen in the distance. The basis of his operator method has been obscure to everyone, but Heaviside himself did not profess to understand this. He merely showed how to work the method, and claimed that it led to correct results. In all cases he was obscure only when he himself thought there was no need further to explain.

Next, as regards the electromagnetic theory, Heaviside's work was to simplify its basis and to develop a mathematical machine specially suitable for use with it, and this machine he applied to physical problems in a way that aroused the admiration of the leaders of electrical science. No greater service can be rendered to any theory than to put its fundamental points in the clearest and most workable form. The greater the theory the greater is the service. This function Heaviside fulfilled so well that he has fixed his mark on Maxwell's theory.

Thirdly, his work on the cable problem, while all directed to one end, necessarily had to surmount one difficulty after another and did a great deal more than lead to the theory of the loading coil. It shows that ordinary telegraphy is but a special form of directed wave telegraphy, and that Heaviside was the first radio telegraphist. The wonderful work of radio engineers refers to the invention and development of the transmitter and of the receiver. In regard to what happens between the two there is more to be found in the writings of Heaviside than elsewhere. Heaviside's analysis in connection with Maxwell's theory, though involving many short cuts, was yet all of standard type, and showed that he could use standard mathematical methods as well as anyone. In the analysis of the cable problem, all things "suffer a 'sea' change into something rich and strange." He threw over conventional forms of mathematics, but never left his problem. The reader could always get help from good textbooks in regard to any difficulty met with in understanding the work on Maxwell's theory, but as regards the cable problem no mathematical library included, until quite recently, any book which is at all helpful in connection with the operator method. Heaviside was a rough-rider exploring new country, not a maker or mere user of railway lines. His work is not to be judged by the difficulty of following in his tracks, but by the news he brought home with him on his return.

We come finally to the debatable conclusion. If mathematicians ever spend on the work of Heaviside as much as 1 per cent of the time they have spent on the work of Fourier, they will find many things which will astonish them. They may not be willing to admit—what I suggest is the fact—that Heaviside was a second Fourier. This much at least is certain. Heaviside, in his search for a solution of a definite problem, explored new ground and turned up many by-products to which he paid no attention, though at times he made a passing comment on their mathematical interest. The result is that his writings are sprinkled with what may be called diamonds in the rough. The mathematician does not appreciate diamonds until they are thoroughly worked up and until they sparkle from every point of view. Many years may pass before these by-products are examined and fitted for admission to the mathematical museum, but sooner or later this work will be done. It will then be clear that the reputation of Heaviside, great as it now is, will become yet greater as time goes on.

APPENDIX 1.

Historical Notes.

An immense advance, both in electrical engineering and in electromagnetic theory, took place during the six years 1882 to 1888. Electrical developments were brought to public notice by the Crystal Palace Exhibition of 1882. This date marks the coming of the good armature due to Gramme, and of the glow lamp due to Edison and Swan. The construction of good dynamos dates from 1886, when two important papers were published dealing with the magnetic circuit. One of these was due to Hopkinson, and the other to Kapp. The former was highly scientific and quite to the point, but not widely understood. The latter contained Kapp's idea of magnetic resistance, now called reluctance, which made the application of known theory crystal clear to the designers of dynamos. The year 1888 marks the coming of alternating-current practice, the date of the Grosvenor Gallery installation, of the running in parallel of transformers and of alternators, and of the generation and distribution of high-tension currents by paper-insulated cables; all developments largely due to the genius of Ferranti.

Maxwell's theory dates from 1864, and his book on "Electricity" from 1873 (first edition), and from 1881

(second edition). It was in 1882 that Heaviside began to write papers on telegraphy from the point of view of the wave theory. That year is important for another reason. Prof. G. F. Fitzgerald tried very hard to devise experiments to prove the existence of Maxwell's waves. He did not succeed, but he did suggest the method afterwards used by Hertz for producing short waves, since he pointed out in 1882 that the discharge of condensers of small capacitance through coils of small inductance, would yield the short waves required. It only needed a spark to convert the suggestion of Fitzgerald into the experiments of Hertz. No one thought of using a spark as an indicator of voltage. Hertz did this in 1886, but his early papers were not printed in Germany until 1887, and his work was not known in England until after Lodge, using sparks imitating lightning flashes, had obtained results similar to those of Hertz. The waves were studied by Hertz as from a ship at sea, and by Lodge as they broke upon the shore. The experiments of Hertz resembled those of light or sound, while those of Lodge formed the earliest examples of wireless telegraphy. Hertz's work was not widely known until Fitzgerald drew public attention to it in an address to the British Association in 1888.

The year 1888 not only marks the beginning of alternating-current engineering. It is as good a date as can be chosen to denote the start of the supremely important researches of J. J. Thomson on the conduction of electricity through gases, which led to the discovery of the electron and to the revolutionary changes in physics. The new theories made possible the invention and development of the heavy-duty mercury-arc rectifier, and also of the thermionic valve by means of which broadcast telephony has been made possible.

Heaviside's share in this development is represented by many papers, written between 1882 and 1885, which made clear, and developed, Maxwell's theory. Poynting and Heaviside were working on this subject quite independently between 1883 and 1885. They covered much the same ground, and ran a close race ending in a dead heat.

In 1883 Heaviside developed some early work of Maxwell on the inductance and resistance of solid conductors and cores for transient currents. He was dealing with what is now known as the skin effect in thick conductors. Lord Rayleigh was doing similar work, and both he and Heaviside were aware that the results, in the light of Maxwell's theory, showed that rapidly changing currents behaved as if they entered, or left, the core through the dielectric, rather than by way of the circuit. The results were as yet devoid of experimental evidence, and few took interest in these investigations. In 1886 the President of this Institution was Prof. Hughes, the inventor of the printing telegraph and also of the microphone. In January of that year he delivered his presidential address, during which he showed and described some wonderful experiments with his induction balance. The effects were most complex, and were due to a combination of many causes. They aroused much attention and criticism. No one was better able to criticize them than Heaviside, who had made a profound study of the induction conditions needed for a true balance. He did criticize Hughes's results, but he was also keenly interested in them, because he realized, and was the first to point out, that they involved the first direct experimental evidence* in favour of Maxwell's wave theory. They preceded the earliest experiments of Hertz.

Heaviside was aware in 1883 that there was no foundation for the common idea that the energy of an electric current travelled round the circuit with the current, but he had not yet worked out from Maxwell's equations how the energy moved. In his "Electrical Papers," vol. 1, p. 377, in a chapter entitled "Transmission of Energy into a Conducting Core," he writes "the direction of maximum transference is therefore perpendicular to the plane containing the directions of the magnetic force and of the current, and its amount proportional to the product of their strengths, and to the sine of the angle between their directions." This was published in the *Electrician* on the 21st June, 1884. He makes no mention of the work of Poynting, though, at a later date, he fully acknowledges Poynting's priority. It is possible, though hardly likely, that he had heard of Poynting's result, but he could not at this date have seen Poynting's paper, since this contains a footnote dated the 19th June, 1884 (see "Collected Papers," p. 184), and could not have been published at the time Heaviside wrote the article.

Poynting's paper containing the theory of the flux of energy, was sent to the Royal Society in December 1883, and was read in January 1884.

During 1884 Heaviside completed his work on Maxwell's equations, and succeeded in putting the fundamental laws into a simple symmetrical form. His results were published, together with his independent proof of the Poynting flux, in January 1885. In February 1885 Poynting read before the Royal Society a second paper which developed the results of the previous (1884) paper. It analyses Maxwell's theory, shows the significance of the cross-cutting law, 4 (c.c.), and, in general, expands his views about the physical nature of the electromagnetic fluxes. It is mainly in regard to the nature of these fluxes that writers on Maxwell's theory differ from one another. It is of special interest to contrast the views of Poynting with those of Heaviside, since each discarded from Maxwell's theory certain points which the other retained.

APPENDIX 2.

The Nature of the Fluxes in Maxwell's Theory.

Faraday's conception of fluxes was physical, while that of Maxwell was mathematical. The wave theory introduced two new scientific concepts: the principle of the localization of energy, and the concept of dielectric currents. Maxwell did not give any clear physical explanation of what he meant by a dielectric current, and, since a conduction current was assumed to flow

* It is evidence of the skin effect that is here referred to. Mr. Evershed in his summary last year of the work of Hughes (*Journal I.E.E*, 1931, vol. 69, p 1248) shows that "Hughes did in fact discover the coherer" and used it as a relay, with a telephone and battery, to detect wave effects at distances up to 300 yards. At the time these results were obtained they were regarded as ordinary inductive effects. In connection with early work on electric waves, reference should be made to the work in 1870 of Bezold who used dust figures, to detect nodes and internodes. Hertz paid a graceful tribute to this work by including a summary of it in his book on electric waves.

along the wire in the direction of H_e, the electric force, it was natural for most students to regard a dielectric current as a vector directed along H_e. This is not really in accordance with the views of Faraday, who always refers to transverse action. Poynting in his 1885 paper adopts a " change in the point of view " in which " induction is regarded as being propagated sideways rather than along the tubes or lines of induction." This amounted to a return to Faraday's physical conception of fluxes. Poynting turns the law 4(x) into the form 4(c.c.), but does not put it into the vector form 4(v).

Heaviside followed Maxwell in treating the fluxes as purely mathematical. It is curious that he does not appear at all to favour the concept of lines or tubes of force. He says (" Electromagnetic Theory," vol. 1, p. 30): " I must, however, wonder at the persistence with which the practicians have stuck to ' the lines,' as they usually term the flux in question." Poynting's concept of a current C was " that C electric induction tubes close in upon the wire per second." This concept of transverse action harmonizes with the vector law 4(v), a law which was in essence realized both by Heaviside and by Poynting, but which, for quite different reasons, neither of them used in analysis.

Heaviside maintained that Maxwell, when establishing his theory, cleared away many cobwebs from the older theories, but did not entirely remove them, since he retained in his equations certain needless quantities expressed in terms of potential functions involving the idea of action at a distance. Hertz agreed with this view, and discarded all remote action effects. In his work on " Electric Waves " (translation by Jones), he says (p. 195), in reference to Maxwell's representation, that " it frequently wavers between the conceptions which Maxwell found in existence, and those to which he arrived "; while as regards the work of Heaviside, he says (p. 196), " from Maxwell's equations he removes the same symbols as myself; and the simplest form which these equations thereby attain is essentially the same as that at which I arrive." Thus Heaviside and Hertz each regard the circuital laws 4(c) as both necessary and sufficient. They gave Heaviside all that he wanted, and this appears to be the chief reason that he made no use of the vector law 4(v). Poynting, however, clung to Maxwell's ideas. Though he arrived at both 4(c) and 4(v), he did not regard either of these laws as sufficient, and used Maxwell's original equations in a slightly modified form.

Heaviside did not pay as much attention as Poynting did to the physical nature of fluxes, but no writer on electromagnetic theory is so insistent on the necessity of regarding H_e, B_e, H_m, and B_m as distinct physical entities. Most writers seem to regard H and B in the ether, if not entirely, yet mainly, as merely two aspects of the same quantity. Poynting, when analysing Maxwell's theory in the 1885 paper, does not even refer to the force-flux law 1. The view of Hertz is clear and drastic. In his book on waves he refers (p. 139) to " a number of auxiliary ideas which render the understanding of Maxwell's theory more difficult, partly for no other reason than that they really possess no meaning, if we exclude the notion of direct action at a distance," and in a footnote he adds " as an example I would mention the idea of a dielectric constant of the ether." He repeats his view of the needless character of this idea on page 196. He writes the circuital laws 4(c) as

$$-A\,\dot{\mathbf{H}}_m = \text{curl}\,\mathbf{H}_e; \quad A\,\dot{\mathbf{H}}_e = \text{curl}\,\mathbf{H}_m;$$

where A is the reciprocal of the velocity of light. He assumes that in the ether $\mathbf{H}_e$ is identical with $\mathbf{B}_e$, and also $\mathbf{H}_m$ with $\mathbf{B}_m$. He measures these in " absolute Gauss's units," so that the square of each has the dimensions of energy with

$$T_e = \frac{1}{8\pi} H_e^2 \qquad T_m = \frac{1}{8\pi} H_m^2$$

Such an aspect of the matter is not in accordance with the views of Heaviside, and really amounts to asserting that all four quantities H_e, H_m, B_e, B_m, have the same physical dimensions.

The needless additions to Maxwell's equations are discussed from a new point of view by Heaviside in his paper printed by the Royal Society in 1892. He criticizes the form of the Poynting flux of energy, and the aspects of the various fluxes occurring in Maxwell's equations. He appears to be discussing the individuality of fluxes when due to distinct energy sources. This subject is one which seems to merit more attention than has hitherto been given to it.

APPENDIX 3.

References.

Bromwich's method is given in the *Proceedings of the London Mathematical Society*, 1916, vol. 15, pp. 401–448, and is developed by H. Jeffreys in *Cambridge Tracts in Mathematics*, No. 23.

Carson's method is described in his book " Electric Circuit Theory and the Operational Calculus," 1929 (McGraw-Hill, New York). It is developed by Balth van der Pol in *Philosophical Magazine*, 1929, vol. 8, p. 861; 1931, vol. 11, p. 368; and 1932, vol. 13, p. 537.

Sumpner's papers are " Heaviside's Fractional Differentiator," *Proceedings of the Physical Society, London*, 1929, vol. 41, p. 404; " Impulse Functions," *Philosophical Magazine*, 1931, vol. 11, p. 345; and " Index Operators," *ibid.*, 1931, vol. 12, p. 201.

The subject of the individuality of fluxes, referred to at the end of Appendix 2, is discussed in " Electromagnetic Waves and Pulses," *Philosophical Magazine*, 1932, vol. 13, p. 1049.